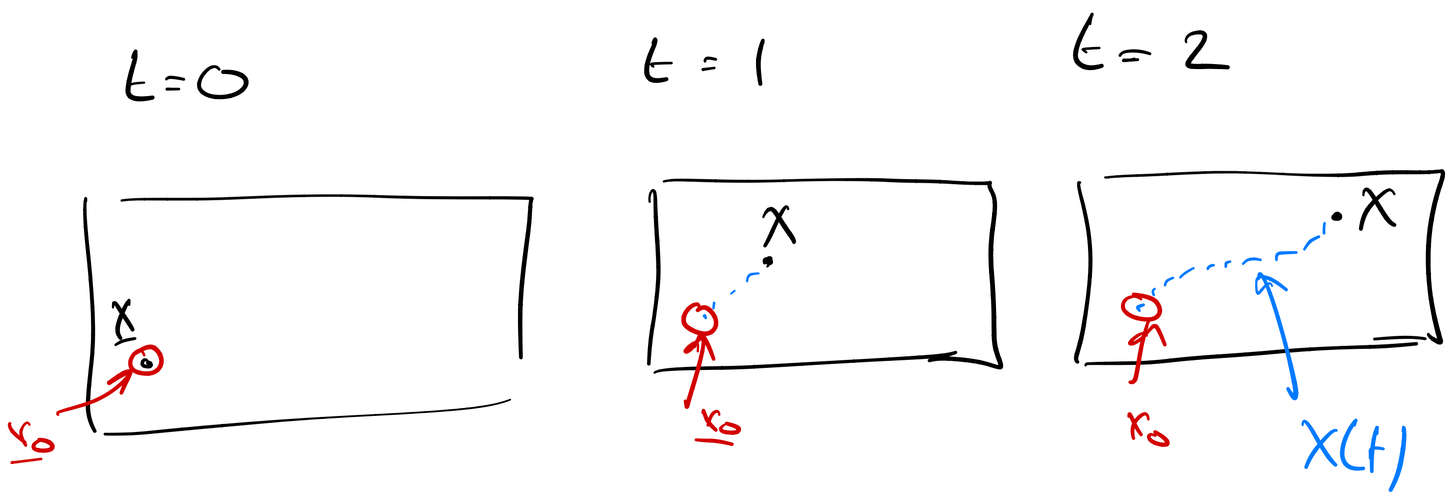
Streamlines, Streaklines and Pathlines

Pathlines $\hat{=}$ Trajectories $\underline{X}(t, \underline{x_0})$

"Motion of the particle in time, given a starting position $\underline{x_0}$"

Ex: photo with a long exposure time of a droplet in water

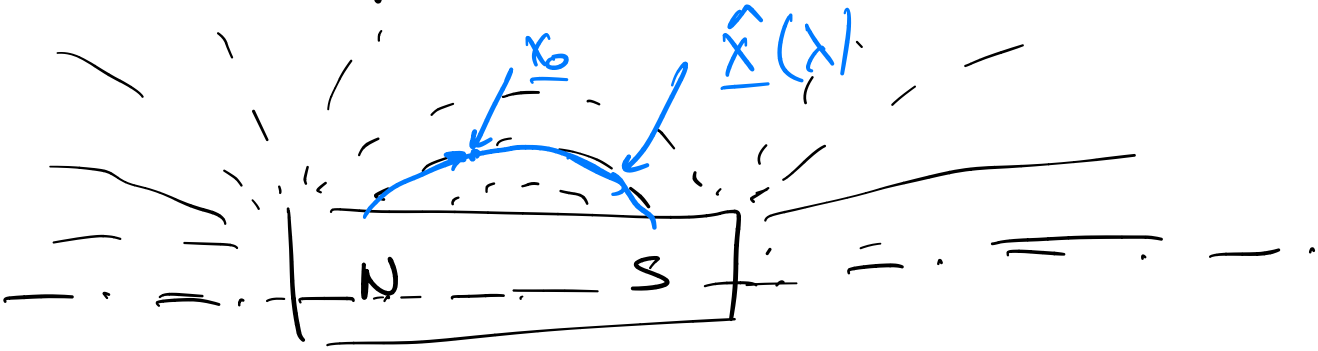


$$\begin{cases} \dfrac{d\underline{X}(t, \underline{x_0})}{dt} = \underline{v}(\underline{X}(t), t) \\ \text{IC: } \underline{X}(t=0; \underline{x_0}) = \underline{x_0} \end{cases}$$

Streamlines:

Fix $t = t^*$ constant.

Ex. Follow the tangents of aluminium filings in water.

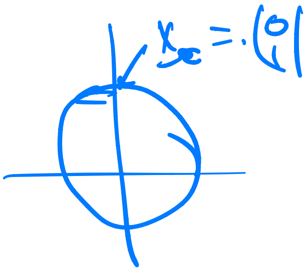


Let $\hat{\underline{x}}(s)$ be a streamline.

$$\frac{d\hat{\underline{x}}(s)}{ds} \parallel \underline{v}(\hat{x}(s); t^*)$$

$$\frac{d\hat{\underline{x}}(s)}{ds} = \lambda \, v(\hat{x}(s), t^*) \qquad (**)$$

Remark: Why λ?



$$\hat{\underline{x}}(s) = \begin{pmatrix} \cos(s) \\ -\sin(s) \end{pmatrix} \to \left\| \frac{d\hat{x}(s)}{ds} \right\| = 1$$

$$\hat{\underline{x}}(\lambda) = \begin{pmatrix} \cos(2\lambda) \\ -\sin(2\lambda) \end{pmatrix} \to \left\| \frac{d\hat{x}(\lambda)}{d\lambda} \right\| = 4$$

$$\lambda := \int h \, ds$$

$$\Rightarrow \quad \underbrace{\frac{d\lambda}{ds} = h}_{} \qquad (*)$$

Insert $(*)$ in $(**)$:

$$\frac{d\hat{X}}{d\lambda} \underbrace{\frac{d\lambda}{ds}}_{= h} = h \, \underline{v}(\hat{X}; t^*)$$

$$\boxed{\frac{d\hat{X}(\lambda)}{d\lambda} = \underline{v}(\hat{X}(\lambda))}$$ Def. of Streamline

multiplying by $\left| \times \frac{d\hat{X}}{d\lambda} \right.$:

we obtain

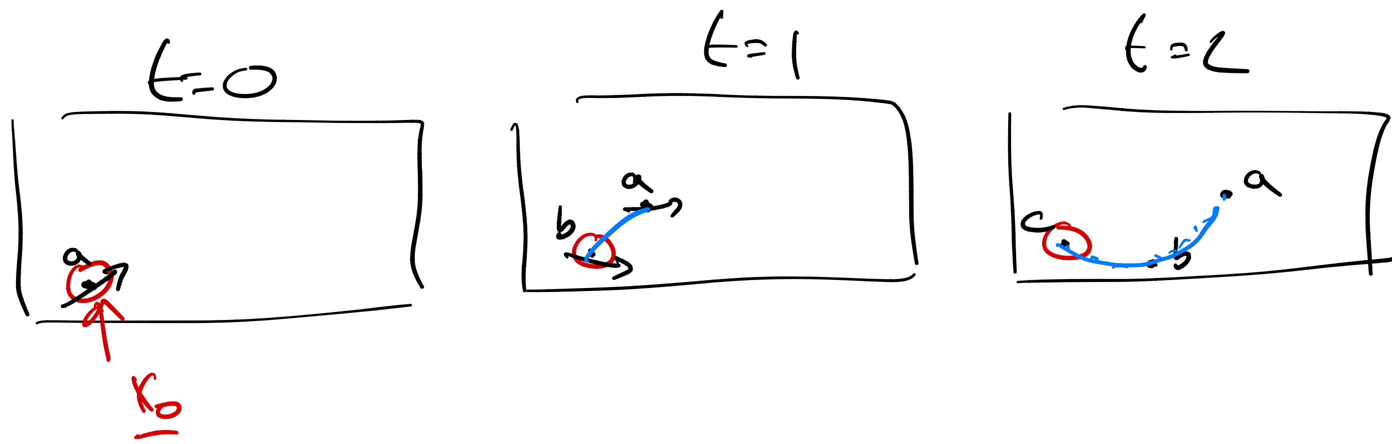
$$\underbrace{\frac{d\hat{X}}{d\lambda} \times \frac{d\hat{X}}{d\lambda}}_{0} = \underline{v}(\hat{X}(\lambda)) \times \frac{d\hat{X}}{d\lambda}$$

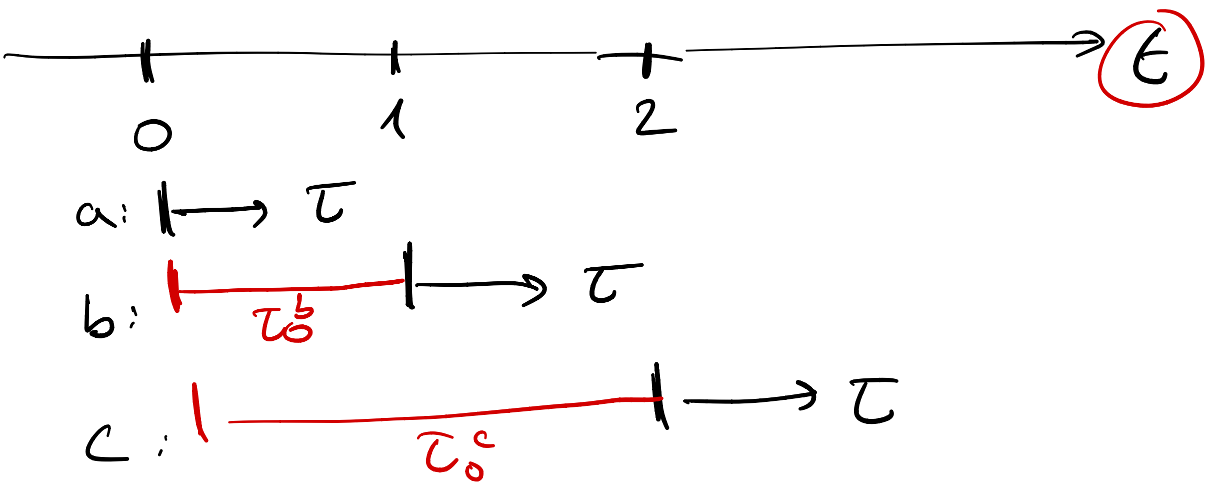
$$\boxed{0 = \underline{v} \times \frac{d\hat{X}}{d\lambda}}$$

Streak lines

Ex: - continously release ink in water at the same position $\underline{r_0}$.

- cigarette smoke





$$t = \tau_0 + \tau$$

τ local time

τ_0 "injection time"

Let $\underline{\tilde{X}}(\tau; \underline{x_0}, \tau_0)$ be a streamline.

The streamline $\underline{\tilde{X}}(\tau; \underline{x_0}, \underline{\tau_0})$ is defined

<u>Def.</u>

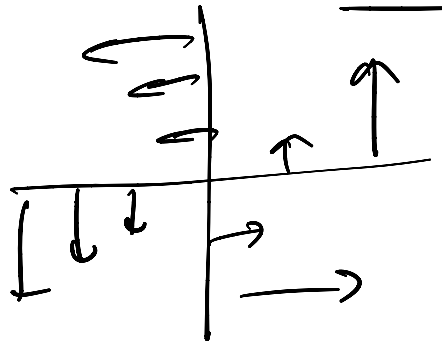
$$\begin{cases} \dfrac{d\tilde{X}}{d\tau} = \underline{v}\left(\tau+\tau_0 ; \underline{\tilde{X}}(\tau)\right) \\ \underline{IC}: \quad \tilde{X}(\tau=0) = \underline{x_0} \end{cases}$$

Exercise: Streamlines / Streaklines / Pathlines

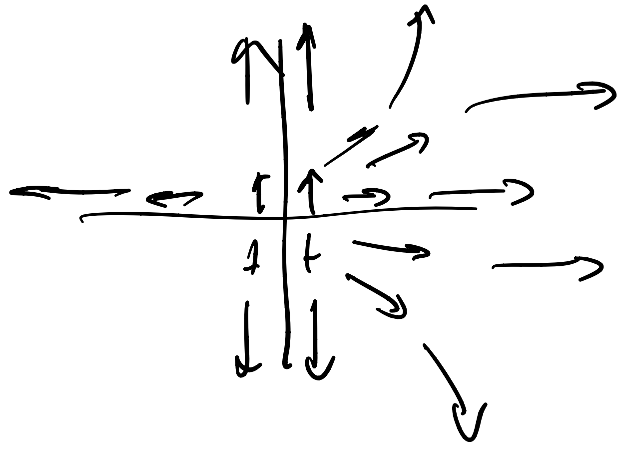
Given $\underline{v}(X(t)) = \begin{pmatrix} tX - y \\ X + tY \end{pmatrix}$

a) Sketch for $t = 0$, $t \gg 1$

$t = 0$



$t \gg 1$



b) Compute the Pathlines:

$$\frac{d\underline{X}(t)}{dt} = \underline{v} \qquad IC: \ \underline{X}(t=0) = \underline{x}_0$$

$$\begin{cases} \dfrac{dX}{dt} = tX - y \\[2ex] \dfrac{dy}{dt} = X + ty \end{cases}$$

Polar coordinates :

$$\begin{cases} r(t=0) = r_0 \\ \phi(t=0) = \phi_0 \end{cases}$$

$$x = r\cos(\phi)$$
$$y = r\sin(\phi)$$

IC : $\begin{pmatrix} r_0 \\ \phi_0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$x_0 = r_0 \cos(\phi_0)$$
$$y_0 = r_0 \sin(\phi_0)$$

$$\frac{dr}{dt}\cos(\phi) - \sin(\phi)\frac{d\phi}{dt} = \varepsilon r\cos(\phi) - r\sin(\phi)$$

$$\frac{dr}{dt}\sin(\phi) + \cos(\phi)\frac{d\phi}{dt} = r\cos(\phi) + \varepsilon r\sin(\phi)$$

$\Leftrightarrow$

$$\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{pmatrix} dr/dt \\ d\phi/dt \end{pmatrix}$$

$$= \begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix} \begin{bmatrix} \varepsilon r \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \frac{dr}{dt} = \varepsilon r \qquad \frac{d\phi}{dt} = 1$$

Integrate

$$r(t) = C e^{t^2/2}$$

$$\phi(t) = C_2 + t$$

IC

$$r(0) = C = r_0$$

$$\phi(0) = C_2 = \phi_0$$

$$r(t) = r_0 e^{t^2/2}$$

$$\phi(t) = t + \phi_0 \iff t = \phi(t) - \phi_0$$

Eliminate t:

$$r(t) = r_0 e^{(\phi(t) - \phi_0)^2/2}$$

$$\boxed{r(\phi) = r_0 e^{(\phi - \phi_0)^2/2}}$$

c) Compute the Streamlines:

Fix $t = t^*$:

$$\frac{d\hat{x}(\lambda)}{d\lambda} = \underline{v}(t^*, \hat{x}(\lambda)) \qquad \text{IC: } \underline{\hat{x}}(\lambda=0) = \underline{x_0}$$

Polar coordinates:

$$\frac{dr}{d\lambda} = t^* r$$

$$\frac{d\phi}{d\lambda} = 1$$

Integrate

$$r(\lambda) = C e^{t^* \lambda}$$

$$\phi(\lambda) = \lambda + C_2$$

IC:

$$r(\lambda) = r_0 e^{t^* \lambda}$$

$$\phi(\lambda) = \lambda + \phi_0$$

Eliminate λ :

$$\lambda = \phi(\lambda) - \phi_0$$

$$r(\phi) = r_0 \, e^{t^* (\phi - \phi_0)}$$

d) Compute the Streaklines:

$\underline{\tilde{X}}(\tau; \underline{x_0}, \tau_0)$ ← Streakline

$\underline{x_0}$ ← "injection point"

τ_0 ← "injection time"

$$\frac{d\underline{\tilde{X}}(\tau)}{d\tau} = \underline{v}(\tau+\tau_0; \tilde{X}(\tau))$$

$\underline{IC}$: $\underline{X}(\tau=0) = \underline{x_0}$

Polar coordinates:

$$\frac{dr}{d\tau} = (\tau+\tau_0)r \qquad \underline{IC}$$

$$\frac{d\phi}{d\tau} = 1$$

$r(\tau=0) = r_0$

$\phi(\tau=0) = \phi_0$

Integration

$$r(\tau) = C \exp\left(1/2\, \tau^2 + \tau_0 \tau \right)$$

$$\phi(\tau) = C_2 + \tau$$

IC

$$C = r_0$$

$$C_2 = \phi_0$$

Eliminate $\tau = \phi(\tau) - \phi_0$

$$r(\phi) = r_0 \exp\left(\frac{1}{2} (\phi - \phi_0)^2 + \tau (\phi - \phi_0) \right)$$

c) Draw pathlines, streamlines and streaklines.